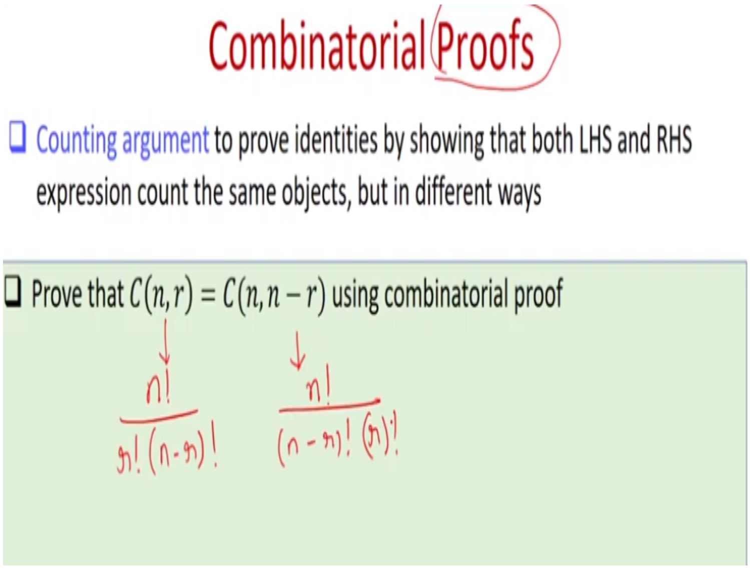

Now let us go to the last topic for today's lecture namely combinatorial proofs and again I am sure that you have studied it during your high school. So what exactly are combinatorial proofs. This are some common proof strategy which we often use in combinatorics. Namely it's a counting argument to prove identities where you have something on the left-hand side and something on your right-hand side and you want to prove mathematically that your expression in the left-hand side and the right-hand side are same.

But to do that we use a counting argument and we prove that the expression in the left-hand side and the expression on the right-hand side count the same number of objects but in different ways. But nowhere in the proof we actually expand our expressions on the left-hand side or right-hand side and show by simplification that left-hand side is same as right-hand side. We do not do that.

That is not the goal of a combinatorial proof. So let us see a very simple combinatorial proof which you must have definitely studied. We will want to prove that the value of $C(n,r)$ is the same as the value of $C(n,n-r)$. Of course one way of doing that is I expand $C(n,r)$ and rewrite it as $\frac{n!}{r!(n-r)!}$. And I expand my right-hand side. And in this case actually both the expressions are same.

So I could have simply said that they are same. But that is not the goal of combinatorial proof. In general when we are giving a combinatorial proof for proving LHS and RHS are same we do not expand or simplify the expressions in the left-hand side and right-hand side. If you do that, that's not a combinatorial proof. You will get 0 marks if you are asked to prove something by combinatorial proof and you end up simplifying expressions.

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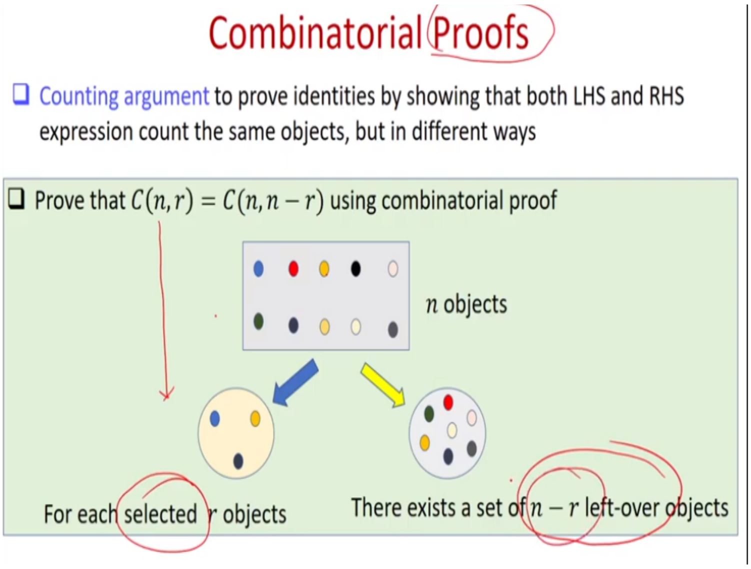

The way we are going to prove this equality using combinatorial proof, is the following. Suppose you are given n objects then your left-hand side is nothing but the number of ways in which you can pick r objects out of those n objects. That's the interpretation of $C(n,r)$ function. Now it turns out for each of the ways in which you can select r objects there is a way of excluding $n-r$ objects.

So I can reinterpret my problem and say that instead of worrying about how many ways I can pick r objects out of n objects, I instead count the number of ways I will decide or the number of ways I will choose the $n-r$ objects which I want to leave. Because once I decide that these are the $n-r$ objects which I am going to leave that automatically gives me the r objects which I will be taking or considering.

So that's why it is easy to see that the LHS expression and RHS expression are same. And we are counting 2 different things here. The left-hand side expression basically counts the number of ways you would have selected the objects. Whereas the right-hand side expression counts the number of ways in which you would have left objects. And there is a mapping. Whatever you left corresponding to that you are left with object which you are picking.

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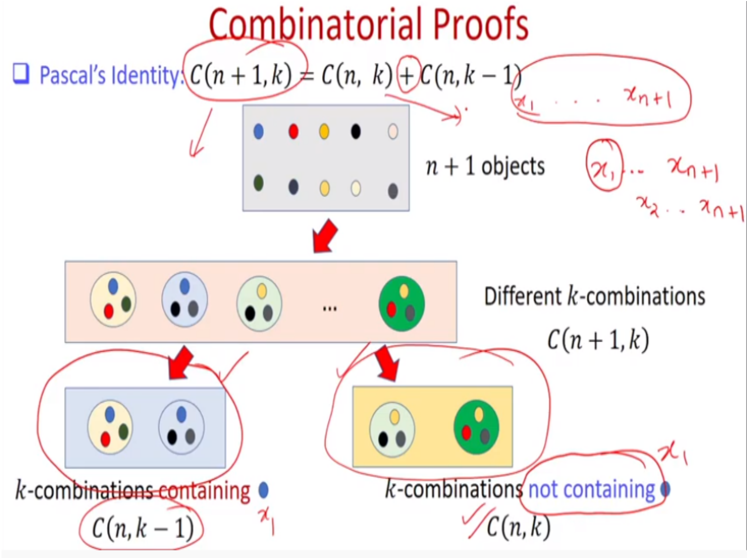

Now let's prove an interesting combinatorial identity using combinatorial proof. This is often called as Pascal's identity. So to prove this consider a collection of $n + 1$ objects. So I am calling those $n + 1$ objects as say x_1 to x_{n+1} they are distinct. Now what is my left-hand side expression? That denotes the total number of k-combinations I can have out of those $n + 1$ objects. I have to pick k objects. I can do that in $C(n + 1, k)$ ways. That is the left-hand side.

Now I have to show that the same thing can be counted by adding these 2 quantities that are there in the right-hand side. How do I do that? So my claim is that different k-combinations that I can have out of x_1 to x_{n+1} can be divided into 2 groups. That will take care of the addition that we have on your right-hand side expression. I have to somehow show that the total number of different k-combinations that I can have can be divided into 2 categories, 2 disjoint categories to be more specific and those 2 disjoint categories are the following.

You consider all k-combinations that you can form out of those $n + 1$ objects where a specific object is always present. Say the object x_1 is always present. And the number of such k-combinations is nothing but $C(n, k - 1)$. Because if the object x_1 is always going to be included in the k objects which you are finally choosing, then you have to worry about how many ways you can pick the remaining $k - 1$ objects out of x_1 to x_n.

So you had x_1 to x_{n+1} so you are always going to choose x_1 that is the category we are right now considering. So now you are left with n objects namely x_2 to x_{n+1} and you have to choose $k-1$ objects out of this remaining n objects which you can do in these many ways. In each such k-combination you include the object x_1 that will give you category 1 of k-combinations.

Whereas category 2 k-combinations are the one where the object x_1 is never included. And it is easy to see that the number of different k-combinations of this category is $C(n, k)$. Because if you are not going to includex_1 then your problem is still to choose k objects and now you are left with only n objects to choose for those k objects. You can choose your k objects only from the collection x_2 to x_{n+1}.

So you are left with only with n possibilities and the number of k-combinations that you can now have in the second category is this. And now if I focus on the total or the different k-combinations that I can have out of this x_1 to x_{n+1}, I can have either a k-combination of category 1 or a k-combination of category 2. Namely in the k combination either, x_1 is there or x_1 is not there. I cannot have any third possible category.

And these 2 categories are disjoint. There is no k-combination where x_1 is present as well as x_1 is absent. So if I sum the total number of k-combinations that I have in category 1 and the number of k-combinations that I have in category 2 that will give me the total number of k-combinations that I can have for a set consisting of $n+1$ objects. And that's precisely your right-hand side. And this is a combinatorial proof because now I have not expanded my left-hand side expression, I have not expanded my right-hand side expression and simplified them. I am just giving a counting argument and proving that LHS and RHS are counting the same things.

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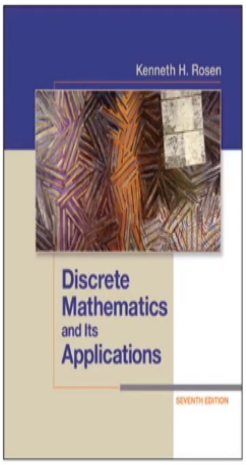
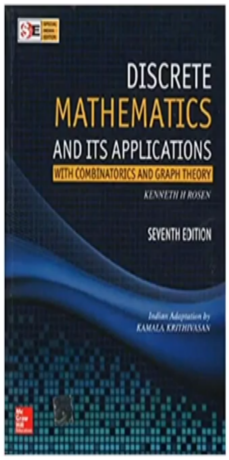
References for Today's Lecture





So that brings me to the end of today's lecture. These are the references used for the today's lecture. Just to summarize, in this lecture we introduced permutations, combinations, we saw the formula for permutations and combinations both with repetitions and without repetition. And we also discussed about combinatorial proofs.